Some Fixed Point Results for Generalized $(a\mathfrak{f} - Q\varphi)$ −Contractive Mappings via Binary Relation in Metric Spaces

Aarju and Anil Kumar
Department of Mathematics,
School of Physical Sciences,
Starex University, Gurugram,
122413, Haryana, India.

Abstract In this manuscript, we expand the concept of generalized altering distance function and introduced a generalized $(\alpha\mathfrak{f} - \beta\varphi)$ −contractive mappings and give fixed point theorems for such contractions in metric spaces. Some existing results of literature are also provided which are direct consequences of our main results.

Keywords complete metric space, fixed point, generalized altering distance function.

Mathematical Subject Classification 2020 47H10, 54H25.

1. Introduction and Preliminaries

Fixed point theory is a branch of mathematics that deals with the study of mappings that have points that remain unchanged under the mapping process. It provides powerful tools and techniques for analyzing the existence, uniqueness, and stability of fixed points in different settings. Fixed point theory has found applications in diverse fields.

One of the most fundamental theorems in fixed point theory is the Banach fixed point theorem [3], also known as the contraction mapping principle. This theorem has numerous applications in areas such as functional analysis, numerical analysis, differential equations, game theory etc. Stefan Banach valuable work has been built by generalizing the metric conditions or by imposing conditions on the metric spaces (see [1-18]).

In this manuscript, we expand the concept of generalized altering distance function and introduced a generalized $(\alpha\mathfrak{f} - \beta\varphi)$ −contractive mappings and give fixed point theorems for such contractions in metric spaces.

Khan *et al.* [11] use a control function (altering distance function) they referred to as a changing distance function allowed them to tackle new fixed point problems.

"**Definition 1.1. [11]** A function $\mathfrak{f} : [0, \infty) \to [0, \infty)$ is called an altering distance function if the following properties are satisfied

   (i) $\mathfrak{f}(0) = 0$ if and only if $t = 0$,

   (ii) $\mathfrak{f}$ is continuous and monotonically non-decreasing."

1

We first recall the auxiliary functions that we shall use effectively.

"**Definition 1.2.** Let K be a set, and let R be a binary relation on K. We say that $P : K \to K$ is an R-preserving mapping if

$ħ, ķ \in K : ħRķ \implies PħRPķ$."

"**Definition 1.3.** Let $\aleph \in \mathbb{N}$. We say that R is $\aleph$-transitive on K if

$ħ_0, ħ_1, \dots, ħ_{\aleph+1} \in K : ħ_i R ħ_{i+1}$, for all $i \in \{0, 1, \dots, \aleph\} \implies ħ_0 R ħ_{\aleph+1}$."

"**Remark 1.4.** Let $\aleph \in \mathbb{N}$. We have:

(i) If R is transitive, then it is $\aleph$-transitive, for all $\aleph \in \mathbb{N}$.

(ii) If R is $\aleph$-transitive, then it is $e\aleph$-transitive, for all $e \in \mathbb{N}$."

"**Definition 1.5.** Let (K, l) be a metric space and R_1, R_2 two binary relations on K. We say that (K, l) is (R_1, R_2)- regular if for sequence $\{ħ_t\}$ in K such that $ħ_t \to ħ \in K$ as $t \to \infty$, and

$$ħ_t R_1 ħ_{t+1}, \qquad ħ_t R_2 ħ_{t+1}, \qquad \text{for all } t \in \mathbb{N},$$

there exists a subsequence $\{ħ_{t(e)}\}$ such that

$$ħ_{t(e)} R_1 ħ, \qquad ħ_{t(e)} R_2 ħ, \qquad \text{for all } e \in \mathbb{N}."$$

"**Definition 1.6.** We say that a subset D of K is (R_1, R_2) −directed if for all $ħ, ķ \in D$, there exists $z \in K$ such that

$$ħR_1z \text{ A } ķR_1z \text{ and } ħR_2z \text{ A } ķR_2z."$$

"**Definition 1.7.** Let K be a set and $\alpha, \beta : K \times K \to [0, +\infty)$ are two mappings. Define two binary relations R_1 and R_2 on K by

$ħ, ķ \in K : ħR_1ķ$ iff $\alpha(ħ, ķ) \leq 1$

and

$ħ, ķ \in K : ħR_2ķ$ iff $\beta(ħ, ķ) \geq 1$."

Berzig and Karapinar [4] introduced $(\alpha ſ, \beta \varphi)$ −contractive mapping as given below:

"**Definition 1.8.** Let (K, l) be a metric space, and let $P : K \to K$ be a given mapping. We say that P is $(\alpha ſ, \beta \varphi)$ −contractive mappings if there exists a pair of generalized distance $(ſ, \varphi)$ such that

$$ſ(l(Pħ, Pķ)) \leq \alpha(ħ, ķ) ſ(l(ħ, ķ)) - \beta(ħ, ķ)\varphi(l(ħ, ķ)), \text{ for all } ħ, ķ \in K,$$

where $\alpha, \beta : K \times K \to [0, +\infty)."$

2. Main Results

Definition 2.1. Consider the pair of the functions $(ſ, \varphi)$ and this pair of the functions is said to be strong generalized altering distance where $ſ, \varphi : [0, \infty) \to [0, \infty)$ if the following conditions hold:

$(C1)$ $\mathfrak{f}(0) = 0$,

$(C2)$ $\mathfrak{f}$ is continuous,

$(C3)$ $\mathfrak{f}$ is non-decreasing,

$(C4)$ $\lim\limits_{t \to \infty} \varphi(x_t) = 0 \implies \lim\limits_{t \to \infty} x_t = 0$.

Popescu in [16] and Moradi and Farajzadeh [15] introduced condition (C4).

Definition 2.2. Let (K, l) be a metric space, and let $P : K \to K$ be a given mapping. We say that P is generalized $(\alpha \mathfrak{f}, \beta \varphi) -$contractive mappings if there exists a pair of generalized distance $(\mathfrak{f}, \varphi)$ such that

$\mathfrak{f}(l(P\mathrm{h}, P\mathrm{k})) \leq \alpha(\mathrm{h}, \mathrm{k})\mathfrak{f}(Z(\mathrm{h}, \mathrm{k})) - \beta(\mathrm{h}, \mathrm{k})\varphi(Z(\mathrm{h}, \mathrm{k})),$ for all $\mathrm{h}, \mathrm{k} \in K,$

(2.1)

where $Z(\mathrm{h}, \mathrm{k}) = \max\left\{l(\mathrm{h}, \mathrm{k}), l(\mathrm{h}, P\mathrm{h}), l(\mathrm{k}, P\mathrm{k}), \frac{l(\mathrm{h}, P\mathrm{h}) + l(\mathrm{k}, P\mathrm{k})}{2}\right\},$

and $\alpha, \beta : K \times K \to [0, \infty)$.

Theorem 2.3. Let (K, l) be a complete metric space, $\mathbb{N}_0 \in \mathbb{N} \cup \{0\}$, and let $P : K \to K$ be generalized $(\alpha \mathfrak{f}, \beta \varphi) -$contractive mapping satisfying the following conditions:

  (i) R_i is $N -$transitive for $i = 1, 2$;

  (ii) P is $R_i -$transitive for $i = 1, 2$;

  (iii) There exists $\mathrm{h}_0 \in K$ such that $\mathrm{h}_0 R_i P\mathrm{h}_0$ for $i = 1, 2$;

  (iv) P is continuous.

Then P has a fixed point, that is, there exists $\mathrm{h} \in K$ such that $P\mathrm{h} = \mathrm{h}$.

Proof Let $\mathrm{h}_0 \in K$ such that $\mathrm{h}_0 R_i P\mathrm{h}_0$ for $i = 1, 2$. Let sequence $\{\mathrm{h}_t\}$ be defined by recursive relation $\mathrm{h}_{t+1} = P\mathrm{h}_t$, for all $t \geq 0$.

If $\mathrm{h}_t = \mathrm{h}_{t+1}$, for some $t \geq 0$, then $\mathrm{h} = \mathrm{h}_t$ is a fixed point of P.

Assume that $\mathrm{h}_t \neq \mathrm{h}_{t+1}$, for all $t \geq 0$.

Form (ii) and (iii), we obtain

$\mathrm{h}_0 R_1 P\mathrm{h}_0 \implies \alpha(\mathrm{h}_0, P\mathrm{h}_0) = \alpha(\mathrm{h}_0, \mathrm{h}_1) \leq 1 \implies \alpha(P\mathrm{h}_0, P\mathrm{h}_1) = \alpha(\mathrm{h}_1, \mathrm{h}_2) \leq 1.$

Similarly, we have

$\mathrm{h}_0 R_2 P\mathrm{h}_0 \implies \beta(\mathrm{h}_0, P\mathrm{h}_0) = \beta(\mathrm{h}_0, \mathrm{h}_1) \geq 1 \implies \beta(P\mathrm{h}_0, P\mathrm{h}_1) = \beta(\mathrm{h}_1, \mathrm{h}_2) \geq 1.$

By Principal of Mathematical induction and using condition (ii), we have

$\alpha(\mathrm{h}_t, \mathrm{h}_{t+1}) \leq 1,$ for all $t \geq 0.$

(2.2)

and, similarly, we have

3

$\beta(h_t, h_{t+1}) \geq 1,$ for   all $t \geq 0.$

(2.3)

Substituting $h = h_t$ and $k = h_{t+1}$ in (2.1), we obtain

$$f(l(Ph_t, Ph_{t+1})) \leq \alpha(h_t, h_{t+1})f(Z(h_t, h_{t+1})) - \beta(h_t, h_{t+1})\varphi(Z(h_t, h_{t+1})),$$

(2.4)

Using (2.2) and (2.3) in inequality (2.4), we obtain

$$f(l(h_{t+1}, h_{t+2})) \leq f(Z(h_t, h_{t+1})) - \varphi(Z(h_t, h_{t+1})),$$

(2.5)

where $Z(h_t, h_{t+1})$

$$= \max\{l(h_t, h_{t+1}), l(h_t, Ph_t), l(h_{t+1}, Ph_{t+1}), \frac{l(h_t, Ph_t) + l(h_{t+1}, Ph_{t+1})}{2}\}$$

$$= \max\{l(h_t, h_{t+1}), l(h_t, h_{t+1}), l(h_{t+1}, h_{t+2}), \frac{l(h_t, h_{t+1}) + l(h_{t+1}, h_{t+2})}{2}\}$$

$$= \max\{l(h_t, h_{t+1}), l(h_{t+1}, h_{t+2}), \frac{l(h_t, h_{t+1}) + l(h_{t+1}, h_{t+2})}{2}\}.$$

Case (i) If $Z(h_t, h_{t+1}) = l(h_t, h_{t+1})$.

From (2.5), we get

$$f(l(h_{t+1}, h_{t+2})) \leq f(l(h_t, h_{t+1})) - \varphi(l(h_t, h_{t+1})),$$

this implies,

$$f(l(h_{t+1}, h_{t+2})) \leq f(l(h_t, h_{t+1})).$$

Since f is non-decreasing function.

Therefore, $l(h_{t+1}, h_{t+2}) \leq l(h_t, h_{t+1})$.

Case (ii) If $Z(h_t, h_{t+1}) = l(h_{t+1}, h_{t+2})$.

From (2.5), we get

$$f(l(h_{t+1}, h_{t+2})) \leq f(l(h_{t+1}, h_{t+2})) - \varphi(l(h_{t+1}, h_{t+2})).$$

Since f is non-decreasing function.

Therefore, above inequality holds only when $l(h_{t+1}, h_{t+2}) = 0$,

this implies,

$h_{t+1} = h_{t+2}$, which is a contradiction.

Hence our supposition was wrong.

Therefore, $Z(h_t, h_{t+1}) \neq l(h_{t+1}, h_{t+2})$.

From above discussed cases, we get $l(h_{t+1}, h_{t+2}) \leq l(h_t, h_{t+1})$.

It shows that sequence $\{l(h_t, h_{t+1})\}$ is monotonically decreasing.

4

From above discussed cases, we also get $Z(h_t, h_{t+1}) = l(h_t, h_{t+1})$. (2.6)

Using (2.6) in (2.5), we obtain

$$\boldsymbol{f}(l(h_{t+1}, h_{t+2})) \leq \boldsymbol{f}(l(h_t, h_{t+1})) - \varphi(l(h_t, h_{t+1})),$$

(2.7)

Thus, there exists $r \in \mathbb{R}_+$ such that $\lim_{t \to \infty} l(h_t, h_{t+1}) = r$.

(2.8)

We will prove that $r = 0$.

Letting $t \to \infty$ in inequality (2.7), we get

$$\lim_{t \to \infty} \boldsymbol{f}(l(h_{t+1}, h_{t+2})) \leq \lim_{t \to \infty} \boldsymbol{f}(l(h_t, h_{t+1})) - \lim_{t \to \infty} \varphi(l(h_t, h_{t+1})),$$

Using (2.8) in above inequality, we get

$$\boldsymbol{f}(r) \leq \boldsymbol{f}(r) - \lim_{t \to \infty} \varphi(l(h_t, h_{t+1})),$$

this implies,

$$\lim_{t \to \infty} \varphi(l(h_t, h_{t+1})) = 0,$$

By using condition (C3), we get

$$\lim_{t \to \infty} l(h_t, h_{t+1}) = 0.$$

(2.9)

On the other hand, by (2.2) and (i), we obtain

$$\alpha(h_u, h_{u+eN+1}) \leq 1 \qquad \text{for} \qquad \text{all} \qquad u, e \geq 0.$$

(2.10)

and, similarly, we have

$$\beta(h_u, h_{u+eN+1}) \geq 1 \qquad \text{for} \qquad \text{all} \qquad u, e \geq 0.$$

(2.11)

Now, substituting $h = h_u$ and $k = h_{u^F}$ in (2.1), where $u' = u + eN + 1$, for some $u, e \geq 0$, we obtain

$$\boldsymbol{f}(l(Ph_u, Ph_{u^F})) \leq \alpha(h_u, h_{u^F})\boldsymbol{f}(Z(h_u, h_{u^F})) - \beta(h_u, h_{u^F})\varphi(Z(h_u, h_{u^F})),$$

(2.12)

where $Z(h_u, h_{u^F})$

$$= \max \left\{ (h_u, h_{u^F}), l(h_u, Ph_u), l(h_{u^F}, Ph_{u^F}), \frac{l(h_u, Ph_u) + l(h_{u^F}, Ph_{u^F})}{2} \right\}$$

$$= \max \left\{ l\ (h_u, h_{u^F}), l(h_u, h_{u+1}), l(h_{u^F}, h_{u^F+1}), \frac{l(h_u, h_{u+1}) + l(h_{u^F}, h_{u+1}^F)}{2} \right\}.$$

Using inequalities (2.10), (2.11) in (2.12), we get

$$\boldsymbol{f}(l(Ph_u, Ph_{u^F})) \le \boldsymbol{f}(Z(h_u, h_{u^F})) - \varphi(Z(h_u, h_{u^F})),$$

(2.13)

where $Z(h_u, h_{u^F}) =$

$$\max\{l(h_u, h_{uF}), l(h_u, h_{u+1}), l(h_{u^F}, h_{u^F+1}), \frac{l(h_u, h_{u+1}) + l(h_{u^F}, h_u{}^F{}_{+1})}{2}\}.$$

Now we have three different subcases.

Subcase (i) If $Z(h_u, h_{u^F}) = l(h_u, h_{u^F})$.

Then inequality (2.13) becomes

$$\boldsymbol{f}(l(h_{u+1}, h_{u^F+1})) \le \boldsymbol{f}(l(h_u, h_{u^F})) - \varphi(l(h_u, h_{u^F})).$$

Similarly, from case (i), we get $l(h_{u+1}, h_{u^F+1}) \le l(h_u, h_{u^F})$.

It shows that sequence $\{l(h_u, h_{u^F})\}$ is monotonically decreasing.

Now repeating the same steps as after equation (2.8), we obtain

$$\lim_{u \to \infty} l(h_u, h_{u^F}) = 0.$$

Subcase (ii) If $Z(h_u, h_{u^F}) = l(h_u, h_{u+1})$.

Then inequality (2.13) becomes

$$\boldsymbol{f}(l(h_{u+1}, h_{u^F+1})) \le \boldsymbol{f}(l(h_u, h_{u+1})) - \varphi(l(h_u, h_{u+1})),$$

Letting $u \to \infty$ and using (2.9), we get

$$\lim_{u \to \infty} \boldsymbol{f}(l(h_{u+1}, h_{u^F+1})) = 0,$$

this implies,

$$\lim_{u \to \infty} l(h_{u+1}, h_{u^F+1}) = 0.$$

Subcase (iii) If $Z(h_u, h_{u^F}) = l(h_{u^F}, h_{u^F+1})$.

Then inequality (2.13) becomes

$$\boldsymbol{f}(l(h_{u+1}, h_{u^F+1})) \le \boldsymbol{f}(l(h_{u^F}, h_{u^F+1})) - \varphi(l(h_{u^F}, h_{u^F+1})).$$

Similarly, from subcase (ii), we get

$$\lim_{u \to \infty} l(h_{u+1}, h_{u^F+1}) = 0.$$

From all above discussed subcases, we conclude that

$$\lim_{u \to \infty} l(h_u, h_{u^F}) = 0.$$

(2.14)

Next, we will prove that $\{h_t\}$ is Cauchy sequence. Suppose, to the contrary, that $\{h_t\}$ is not a Cauchy sequence.

6

Then there is $\in > 0$ and sequences $\{u(e)\}$ and $\{t(e)\}$ such that, for all positive integers e, we have

$$t(e) > u(e) > e, \qquad l(h_{u(e)}, h_{t(e)}) \geq \in \qquad \text{and} \qquad l(h_{u(e)}, h_{t(e)-1}) < \in$$

(2.15)

Then we have

$$\in \leq l(h_{u(e)}, h_{t(e)}) \leq l(h_{u(e)}, h_{t(e)-1}) + l(h_{t(e)-1}, h_{t(e)}) < \in + l(h_{t(e)-1}, h_{t(e)})$$

Letting $e \to \infty$ and using (2.9), we get

$$\lim_{e \to \infty} l(h_{u(e)}, h_{t(e)}) = \in. \tag{2.16}$$

Furthermore, for each $e \geq 0$, there exists $\mu_e, 5_e > 0$ such that $u'(e) = u(e) + N\mu_{e+1} + 1 = t(e) + 5_e$.

$$\in \leq l(h_{u(e)}, h_{u^F(e)}) \leq l(h_{u(e)}, h_{t(e)-1}) + \sum_{i=t(e)-1}^{u^F(e)-1} l(h_i, h_{i+1}) < \in + \sum_{i=t(e)-1}^{u^F(e)-1} l(h_i, h_{i+1})$$

Again, letting $e \to \infty$ and using (2.9), we get

$$\lim_{e \to \infty} l(h_{u(e)}, h_{u^F(e)}) = \in.$$

(2.17)

Again

$$l(h_{u(e)}, h_{u^F(e)}) \leq l(h_{u(e)}, h_{u(e)-1}) + l(h_{u(e)-1}, h_{u^F(e)-1}) + l(h_{u^F(e)-1}, h_{u^F(e)}),$$

$$l(h_{u(e)-1}, h_{u^F(e)-1}) \leq l(h_{u(e)-1}, h_{u(e)}) + l(h_{u(e)}, h_{u^F(e)}) + l(h_{u^F(e)}, h_{u^F(e)-1})$$

Letting $e \to \infty$ in above inequalities, using (2.9), (2.14) and (2.17), we get

$$\lim_{e \to \infty} l(h_{u(e)-1}, h_{u^F(e)-1}) = \in.$$

(2.18)

Substituting $h = h_{u(e)-1}$ and $k = h_{u^F(e)-1}$ in (2.1), we have

$$\mathcal{F}(l(Ph_{u(e)-1}, Ph_{u^F(e)-1})) \leq \alpha(h_{u(e)-1}, h_{u^F(e)-1})\mathcal{F}(Z(h_{u(e)-1}, h_{u^F(e)-1})) -$$

$$\beta(h_{u(e)-1}, h_{u^F(e)-1})\varphi\ (Z(h_{u(e)-1}, h_{u^F(e)-1}))$$

Using (2.10) and (2.11) in above inequality, we get

$$\mathfrak{f}\ (l(h_{u(e)}, h_{u^F(e)})) \leq \mathfrak{f}\ (Z(h_{u(e)-1}, h_{u^F(e)-1})) - \varphi\ (Z(h_{u(e)-1}, h_{u^F(e)-1})),$$

(2.19)

where $Z(h_{u(e)-1}, h_{u^F(e)-1})$

7

$$= \max\left\{ \begin{array}{c} l(h_{u(e)-1}, h_{u^F(e)-1}),\ l(h_{u(e)-1}, Ph_{u(e)-1}),\ l(h_{u^F(e)-1}, Ph_{u^F(e)-1}), \\ \dfrac{l(h_{u(e)-1}, Ph_{u(e)-1}) + l(h_{u^F(e)-1}, Ph_{u^F(e)-1})}{2} \end{array} \right\}$$

$$= \max\left\{ \begin{array}{c} l(h_{u(e)-1}, h_{u^F(e)-1}),\ l(h_{u(e)-1}, h_{u(e)}),\ l(h_{u^F(e)-1}, h_{u^F(e)}), \\ \dfrac{l(h_{u(e)-1},h_{u(e)})+l(h_{u^F(e)-1},h_{u^F(e)})}{2} \end{array} \right\}.$$

Subsubcase (i) If $Z(h_{u(e)-1}, h_{u^F(e)-1}) = l(h_{u(e)-1}, h_{u^F(e)-1})$.

Then inequality (2.19) becomes

$$f\left(l(h_{u(e)}, h_{u^F(e)})\right) \leq f\left(l(h_{u(e)-1}, h_{u^F(e)-1})\right) - \varphi\left(l(h_{u(e)-1}, h_{u^F(e)-1})\right),$$

Letting $e \to \infty$ in above inequality, using (2.17), (2.18) and the continuity of f and φ, we get

$$f(\in) \leq f(\in) - \lim_{e \to \infty} \varphi\left(l(h_{u(e)-1}, h_{u^F(e)-1})\right).$$

Using the condition (C3), we conclude that $\in = 0$.

Subsubcase (ii) If $(h_{u(e)-1}, h_{u^F(e)-1}) = l(h_{u(e)-1}, h_{u(e)})$.

Then inequality (2.19) becomes

$$f\left(l(h_{u(e)}, h_{u^F(e)})\right) \leq f\left(l(h_{u(e)-1}, h_{u(e)})\right) - \varphi\left(l(h_{u(e)-1}, h_{u(e)})\right),$$

Letting $e \to \infty$ in above inequality, using (2.9) and the continuity of f and φ, we get

$$f(\in) \leq f(0) - \varphi(0) = 0,$$

this implies,

$$f(\in) = 0 \Longrightarrow \in = 0.$$

Subsubcase (iii) If $Z(h_{u(e)-1}, h_{u^F(e)-1}) = l(h_{u^F(e)-1}, h_{u^F(e)})$.

Then inequality (2.19) becomes

$$f\left(l(h_{u(e)}, h_{u^F(e)})\right) \leq f\left(l(h_{u^F(e)-1}, h_{u^F(e)})\right) - \varphi\left(l(h_{u^F(e)-1}, h_{u^F(e)})\right).$$

Letting $e \to \infty$ in above inequality, using (2.9) and the continuity of f and φ, we get

$$f(\in) \leq f(0) - \varphi(0) = 0,$$

this implies,

$$f(\in) = 0 \Longrightarrow \in = 0.$$

From all above discussed three subsubcases, we find $\in = 0$, which is a contraction with $\in > 0$.

Hence our supposition was wrong. Hence, therefore $\{h_t\}$ is a Cauchy sequence.

Since (K, l) is a complete metric space, then there is $h \in K$ such that $\lim\limits_{t \to \infty} h_t = h$.

Since P is continuous, then we have

8

$$ɦ = \lim_{t\to\infty} ɦ_{t+1} = \lim_{t\to\infty} Pɦ_t = Pɦ.$$

Due to uniqueness of the limit, we derive that $Pɦ = ɦ$, that is, $ɦ$ is a fixed point P.

Theorem 2.4 In Theorem 2.3, if we replace the continuity of P by the (R_1, R_2) −regularity of (K, l), then the conclusion of Theorem 2.3 holds.

Proof Following the proof of Theorem 2.3, we know that the sequence $\{ɦ_t\}$ defined by $ɦ_{t+1} = Pɦ_t$ for all $t \geq 0$, converges to some $ɦ \in K$. Since (K, l) is a complete metric space, then there exists $ɦ \in K$ such that $ɦ_t \to ɦ$ as $t \to \infty$.

Furthermore, the sequence $\{ɦ_t\}$ satisfies (2.2) and (2.3), that is,

$$ɦ_t R_1 ɦ_{t+1}, \qquad ɦ_t R_2 ɦ_{t+1}, \quad \text{for all } t \in \mathbb{N}.$$

Now, since is (R_1, R_2) −regular, then there exists a subsequence $\{ɦ_{t(e)}\}$ of $\{ɦ_t\}$ such that $ɦ_{t(e)} R_1 ɦ$, that is, $\alpha(ɦ_{t(e)}, ɦ) \leq 1$ and $ɦ_{t(e)} R_2 ɦ$, that is, $\beta(ɦ_{t(e)}, ɦ) \geq 1$, for all e. (2.20)

Substituting $ɦ = ɦ_{t(e)}$ and $ķ = ɦ$, in (2.1), we obtain

$$\mathbf{f}(l(Pɦ_{t(e)}, Pɦ)) \leq \alpha(ɦ_{t(e)}, ɦ)\mathbf{f}(Z(ɦ_{t(e)}, ɦ)) - \beta(ɦ_{t(e)}, ɦ)\varphi(Z(ɦ_{t(e)}, ɦ)), \text{for all e.}$$

Using (2.20) in above inequality, we obtain

$$\mathfrak{f}(l(Pɦ_{t(e)}, Pɦ)) \leq \mathfrak{f}(Z(ɦ_{t(e)}, ɦ)) - \varphi(Z(ɦ_{t(e)}, ɦ)), \qquad \text{for} \qquad \text{all} \qquad \text{e,}$$

(2.21)

where $Z(ɦ_{t(e)}, ɦ) = \max\{l(ɦ_{t(e)}, ɦ), l(ɦ_{t(e)}, ɦ_{t(e)+1}), l(ɦ, Pɦ), \frac{l(ɦ_{t(e)}, Pɦ_{t(e)})+l(ɦ,Pɦ)}{2}\}$

Case (i) If $Z(ɦ_{t(e)}, ɦ) = l(ɦ_{t(e)}, ɦ)$.

Then inequality (2.21) becomes

$$\mathfrak{f}(l(ɦ_{t(e)+1}, Pɦ)) \leq \mathfrak{f}(l(ɦ_{t(e)}, ɦ)) - \varphi(l(ɦ_{t(e)}, ɦ)), \text{for all e.}$$

this implies,

$$\mathfrak{f}(l(ɦ_{t(e)+1}, Pɦ)) \leq \mathfrak{f}(l(ɦ_{t(e)}, ɦ)), \text{for all e.}$$

Since $\mathfrak{f}$ is non-decreasing function. Therefore,

$$l(ɦ_{t(e)+1}, Pɦ) \leq l(ɦ_{t(e)}, ɦ), \text{for all e.}$$

Letting $e \to \infty$ in above inequality, we obtain

$$l(ɦ, Pɦ) = 0 \implies ɦ = Pɦ.$$

Case (ii) If $Z(ɦ_{t(e)}, ɦ) = l(ɦ_{t(e)}, ɦ_{t(e)+1})$.

Then inequality (2.21) becomes

$$\mathfrak{f}(l(ɦ_{t(e)+1}, Pɦ)) \leq \mathfrak{f}(l(ɦ_{t(e)}, ɦ_{t(e)+1})) - \varphi(l(ɦ_{t(e)}, ɦ_{t(e)+1})), \text{for all e.}$$

9

Letting $e \to \infty$ in above inequality, we obtain

$$l(\hbar, P\hbar) = 0 \implies \hbar = P\hbar.$$

Case (iii) If $Z(\hbar_{t(e)}, \hbar) = l(\hbar, P\hbar)$.

Then inequality (2.21) becomes

$$\mathcal{f}(l(\hbar_{t(e)+1}, \ P\hbar)) \leq \mathcal{f}(l(\hbar, P\hbar)) - \varphi(l(\hbar, P\hbar)), \text{ for all e.}$$

Letting $e \to \infty$ in above inequality, and the continuity of f and φ, we get

$$\mathcal{f}(l(\hbar, \ P\hbar)) \leq \mathcal{f}(l(\hbar, P\hbar)) - \varphi(l(\hbar, P\hbar)), \text{ for all e.}$$

$$\lim_{e \to \infty} \varphi(l(\hbar, P\hbar)) = 0.$$

By using condition (C3), we get

$$l(\hbar, P\hbar) = 0 \implies \hbar = P\hbar.$$

From all above discussed three cases, we get $P\hbar = \hbar$.

Theorem 2.5. Adding to the hypotheses of Theorem 2.3 (respectively, Theorem 2.4) that K is (R_1, R_2) −directed, we obtain uniqueness of the fixed point of P.

Proof Suppose that $\hbar$ and $\Bbbk$ are two fixed points of P. Since K is (R_1, R_2) −directed, there exists $z \in K$ such that

$$\alpha(\hbar, z) \leq 1, \alpha(\Bbbk, z) \leq 1.$$

(2.22)

and

$$\beta(\hbar, z) \geq 1, \beta(\Bbbk, z) \geq 1.$$

(2.23)

Since P is R_i −preserving for $i = 1, 2$, from (2.22) and (2.23), we get

$$\alpha(\hbar, P^t \ z) \leq 1, \alpha(\Bbbk, P^t z) \leq 1, \qquad \text{for} \qquad \text{all} \qquad t \geq 0.$$

(2.24)

and

$$\beta(\hbar, P^t z) \geq 1, \beta(\Bbbk, P^t \ z) \geq 1, \qquad \text{for} \qquad \text{all} \qquad t \geq 0.$$

(2.25)

Substituting $\hbar = h, \Bbbk = P^t z$ in (2.1), we have

$$\mathcal{f}(l(P\hbar, P(P^t z))) \leq \ \alpha(\hbar, P^t z)\mathcal{f}(Z(\hbar, P^t z)) - \beta(\hbar, P^t z)\varphi(Z(\hbar, P^t z)).$$

(2.26)

Using (2.24), (2.25) and (2.1), we obtain

$$\mathcal{f}(l(\hbar, P^{t+1} z)) \leq \mathcal{f}(Z(\hbar, P^t z)) - \varphi(Z(\hbar, P^t z)),$$

(2.27)

10

where $Z(h, P^t z) = \max\{l(h, P^t z), l(h, Ph), l(P^t z, P^{t+1} z), \dfrac{l(h, Ph) + l(P^t z, P^{t+1} z)}{2}\}$

$$= \max\{l(h, P^t z), l(P^t z, P^{t+1} z), \dfrac{l(P^t z, P^{t+1} z)}{2}\}.$$

Case (i) If $Z(h, P^t z) = l(h, P^t z)$.

Then inequality (2.27) becomes

$\boldsymbol{f}(l(h, P^{t+1} z)) \leq \boldsymbol{f}(l(h, P^t z)) - \varphi(l(h, P^t z))$, for all $t \geq 0$.

(2.28)

Since f is non-decreasing function.

$l(h, P^{t+1} z) \leq l(h, P^t z)$, for all $t \geq 0$.

It follows that the sequence $\{l(h, P^{t+1} z)\}$ is decreasing. Thus there exists $r \geq 0$ such that

$$\lim_{t \to \infty} l(h, P^{t+1} z) = r.$$

We claim that $r = 0$.

Letting $t \to \infty$ in (2.28), we get

$$\boldsymbol{f}(r) \leq \boldsymbol{f}(r) - \lim_{t \to \infty} \varphi(l(h, P^t z)),$$

this implies,

$$\lim_{t \to \infty} \varphi(l(h, P^t z)) = 0.$$

(2.29)

By condition (C3), we obtain

$$\lim_{t \to \infty} l(h, P^t z) = 0.$$

(2.30)

Similarly, we get

$$\lim_{t \to \infty} l(k, P^t z) = 0.$$

Using (2.29) and (2.30), the uniqueness of the limit gives us $h = k$.

Case (ii) If $Z(h, P^t z) = l(P^t z, P^{t+1} z)$

Then inequality (2.27) becomes

$\boldsymbol{f}(l(h, P^{t+1} z)) \leq \boldsymbol{f}(l(P^t z, P^{t+1} z)) - \varphi(l(P^t z, P^{t+1} z))$, for all $t \geq 0$

(2.31)

Letting $t \to \infty$ in (2.31), we get

$$\lim_{t \to \infty} \boldsymbol{f}(l(h, P^t z)) = 0,$$

this implies,

11

$$\lim_{t\to\infty} l(\textrm{h}, P^t z) = 0.$$

(2.32)

Similarly, we get

$$\lim_{t\to\infty} l(\textrm{k}, P^t z) = 0.$$

(2.33)

Using (2.32) and (2.33), the uniqueness of the limit gives us $\textrm{h} = \textrm{k}$.

Corollary 2.6. Let (K, l) be a complete metric space, and let $P : K \to K$ be a given mapping such that if there exists a pair of generalized distance $(\textbf{f}, \varphi)$ such that

$\textbf{f}(l(P\textrm{h}, P\textrm{k})) \le \alpha(\textrm{h}, \textrm{k})\textbf{f}(l(\textrm{h}, \textrm{k})) - \beta(\textrm{h}, \textrm{k})\varphi(l(\textrm{h}, \textrm{k})),$ for all $\textrm{h}, \textrm{k} \in K$,

And $\alpha, \beta : K \times K \to [0, \infty)$.

Suppose

(i) R_i is N −transitive for i $= 1, 2$;

(ii) P is R_i −transitive for i $= 1, 2$;

(iii) There exists $\textrm{h}_0 \in K$ such that $\textrm{h}_0 R_i P\textrm{h}_0$ for i $= 1, 2$;

(iv) P is continuous.

Then P has a fixed point, that is, there exists $\textrm{h} \in K$ such that $P\textrm{h} = \textrm{h}$.

Proof Taking $Z(\textrm{h}, \textrm{k}) = l(\textrm{h}, \textrm{k})$ in Theorem 2.3 to get the proof.

Corollary 2.5. In Corollary 2.6, if we replace the continuity of P by the (R_1, R_2) −regularity of (K, l), then the conclusion of Corollary 2.6 holds.

References

1. Alsamir H., Noorani M. S., Shatanawi W., and Shaddad F., "GeGeneralized Berinde-type (5, £, P, θ) contractive mappings in B-metric spaces with an application," *J. Math. Anal.,* **6**(2016), 1-12.

2. Ansari A. H., Shatanawi W., Kurdi A., and Maniu G., "Best proximity points in complete metric spaces with (P)-property via C-class functions," *J. Math. Anal.,* **7**(2016), 54-67.

3. Banach S., "Sur les opérations dans les ensembles abstraits et leur application aux équations integrals," *Fundam. Math.,* **3**(1)(1922), 133–181.

4. Berzig M., and Karapinar E., "Fixed point results for$(\alpha f − \beta \varphi)$-contractive mappings for a generalized altering distance", *Fixed Point Theory and Applications,* **2013**(2013), 18 pages.

12

5. Dutta P.N., and Choudhury B.S., "A generalisation of contraction principle in metric spaces", *Fixed Point Theory and Applications*, **2008**(2008), 1-8.

6. Karapinar E., and Fugla A. Discussion on the hybrid Jaggi-Meir-Keeler type contractions. **7**(7), 12702-12717 (2022).

7. Karapinar E., "Fixed point theory for cyclic weak φ-contraction", *Appl. Math. Lett.* **24**(6)(2011), 822-825.

8. Karapinar E., and Sadarangani K., "Fixed point theory for cyclic $(\varphi - f)$-contractions", *Fixed Point Theory and Applications*, **2011**(2011), 69.

9. Karapinar E., and Samet B., "Generalized $(\alpha - f)$contractive type mappings and related fixed point theorems with applications", *Abstr. Appl. Anal.* **2012**(2012).

10. Khan M. S., "A fixed point theorem for metric spaces," *Rend. Ist. Mat. Univ. Trieste,* **8**(1976), 69-72.

11. Khan M.S., Swaleh M., and Sessa S., "Fixed point theorems by altering distances between the points", *Bulletin of the Australian Mathematical Society*, **30**(1)(1984), 1–9.

12. Moradi S., and Farajzadeh A., "On the fixed point $(f - \varphi)$-weak contraction mappings", *Appl. Math. Lett.* **25**(10)(2012), 1257-1262.

13. Popescu O., "Fixed points for (f, φ)-weak contractions", *Appl. Math. Lett.* **24**(1)(2011), 1-4.

14. Rhoades B.E., "Some theorem on weakly contractive maps", *Nonlinear Analysis*: *Theory, Methods and Applications,* **31**(2001), 71-81.

15. Samet B., and Turinici M., "Fixed point theorems on a metric space endowed with an arbitrary binary relation applications", *Commun. Math. Anal.,* **13**(2)(2012), 82-97.

16. Samet B., Vetro C., and Vetro P., "Fixed point theorems for $\alpha - f$- contractive type mappings," *Nonlinear Anal.,* **75**(2012), 2154-2165.

17. Shatanawi W., Noorani M. S., Alsamir H., and Bataihah A., "Fixed and common fixed point theorems in partially ordered quasi-metric spaces," *J. Math. Computer. Sci.,* **16**(2016), 516-528.

18. Shatanawi W., and Postolache M., "Common fixed point results for mappings under nonlinear contraction of cyclic form in ordered metric spaces," *Fixed Point Theory Appl.,* **2013**(2013), 13 pages.

13