# Fixed Point Theorems Using Binary Relation in Soft Metric Spaces

[1]Aarju and [1,*]Anil Kumar
[1]Department of Mathematics,
School of Physical Sciences,
Starex University, Gurugram,
122413, Haryana, India.
(*Corresponding Author)

**Abstract:** In this paper, our aim is to prove some fixed point results in complete soft metric spaces by introducing the binary relation in soft metric spaces. Some examples are also given to illustrate our result.

**Keywords** Binary Relations, Complete Soft Metric Spaces, Soft Contraction Mappings, Soft Metric Spaces.

**Mathematical Subject Classification 2020** 47H10, 54H25.

## 1. Introduction

Using traditional classical approaches is not always successful since there are many kinds of uncertainties and ambiguity, either inherent in the data or caused by the mathematical techniques employed to solve the model's complex problems. Probability theory, fuzzy set theory [17], intuitionistic fuzzy set theory [3], vague set theory [9], rough set theory [14], and interval of mathematics ([4], [10]) are some mathematical theories that can be used to deal with uncertainty. The methods employed to quantify objects in these theories have limitations.

To overcome this, in 1999, Molodtsov proposed [12], soft sets as a mathematical tool for dealing with uncertainties associated with real world data-based problems. It offers sufficient tools to handle data uncertainties and display it in a usable manner. This theory has successfully addressed the issue of inadequate parameters.

Alam and Imdad [2] introduced a version of the Banach contraction principle using a relation-theoretic approach. This formulation extends and generalizes several well-known fixed point theorems based on order theory.

Ahmadullah *et al.* [1] has explored several concepts of binary relation for non-linear contractions.

Molodtsov *et al.* [13] has developed applications of soft set and soft theory in various fields, such as smoothness of functions, game theory, operations research etc.

Maji *et al.* [11] has introduced and explored several fundamental concepts in soft set theory, providing a comprehensive understanding of its principles.

Cagman and Enginoglu [5] has conducted research on the products of soft sets and uni-int decision functions.

Sezgin and Atagun [15] has extended the theoretical aspects of operations on soft sets by exploring the De Morgan's laws in soft set theory with respect to different operations.

Shabir and Naz [16] pioneered the investigation of soft topological spaces and showed that a soft topological space corresponds to a parameterized family of topological spaces.

Das and Samanta [6-7] introduced the notion of soft real sets and soft real numbers and studied various properties of soft real sets and numbers. After that, they introduced a notion of a soft metric space which is defined over an initial universe with fixed set of parameters.

In this paper, the contraction condition in soft metric spaces is comparatively weaker than the usual contraction by introducing binary relation in soft metric spaces, as it is required to hold only on those elements which are related under the underlying relation, rather than applying to the entire space.

"**Definition 1.1. [12]** Let $H$ be an initial universe set and $K$ be a non-empty set of parameters. Let $P(H)$ denotes the power set of $H$. A pair $(F, K)$ is called a soft set over $H$, where $F$ is mapping given by $F : K \to P(H)$.

In other words, soft set over $H$ is a parameterized family of subsets of universe $H$. For $a \subset K, F(a)$ may be considered as the set of $a$-approximate elements of the soft set $(F, K)$, or as the set of $a$-approximate elements of the soft set."

"**Definition 1.2. [11]** A soft set $(F, K)$ over $H$ is said to be a null soft set denoted by $\Phi$ if for all $a \in K$, $F(a) = \Phi$."

"**Definition 1.3. [11]** A soft set $(F, K)$ over $H$ is said to be an absolute soft set denoted by $\widetilde{H}$ if for all $\in K$, $F(a) = H$."

"**Definition 1.4. [8]** Let $(F, A)$ and $(G, B)$ be two soft sets over $H$. We say that $(F, A)$ is a soft subset of $(G, B)$ and denote it by $(F, A) \mathrel{\tilde{\subset}} (G, B)$ if

(i) $A \subset B,$ and

(ii) $F(a) \subseteq G(a)$, for all $a \in A$.

$(F, A)$ is said to be a soft super set of $(G, B)$, if $(G, B)$ is a soft subset of $(F, A)$. We denote it by $(F, A) \mathrel{\tilde{\supset}} (G, B)$."

"**Definition 1.5. [6]** Let $H$ be a non-empty set and $K$ be a non-empty set of parameters. Then a function $\mathsf{s} : K \to H$ is said to be a soft element of $H$. A soft element $\mathsf{s}$ of $H$ is said to belong to a soft set $F$ of $H$, denoted by $\mathsf{s} \,\tilde{\in}\, F$, if $\mathsf{s}(\mathrm{e}) \in F(\mathrm{e})$, for all $\mathrm{e} \in K$.

In that case $\mathsf{s}$ is also said to be a soft element of the soft set $F$. Thus, for a soft set $F$ of $H$ with respect to the index set $K$, we have $F(\mathrm{e}) = \{\mathsf{s}(\mathrm{e}), \mathsf{s} \,\tilde{\in}\, F\}, \mathrm{e} \in K$.

It is to be noted that every singleton soft set (a soft set $(F, K)$ for which $F(\mathrm{e})$ is a singleton set, for all $\mathrm{e} \in K$) can be identified with a soft element by simply identifying the singleton set with element that contains for all $\mathrm{e} \in K$."

"**Definition 1.6. [6]** Let $\mathbb{R}$ be the set of real numbers, $\&(\mathbb{R})$be the collection of all non-empty bounded subsets of $\mathbb{R}$ and $K$ be taken as set of parameters. Then a mapping $F : K \to \&(\mathbb{R})$ is called a soft real set. It is denoted by $(F, K)$. If $(F, K)$ is a singleton soft set, then it will be called a soft real number and denoted by $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ etc. Here $\tilde{r}$, $\tilde{s}$, $\tilde{t}$ will denote a particular type of soft real numbers such that $\tilde{r}(a) = r$, for all $a \in K$. $\tilde{0}$ and $\tilde{1}$ are the soft real numbers where $\tilde{0}(a) = 0$, $\tilde{1}(a) = 1$ for all $a \in K$, respectively."

"**Definition 1.7. [6]** Let $\tilde{r}$, $\tilde{s}$ be two soft real numbers, then the following statements hold

(i) $\tilde{r} \,\tilde{\leq}\, \tilde{s}$, if $\tilde{r}(a) \leq \tilde{s}(a)$, for all $a \in K$,

(ii) $\tilde{r} \,\tilde{\geq}\, \tilde{s}$, if $\tilde{r}(a) \geq \tilde{s}(a)$, for all $a \in K$,

(iii) $\tilde{r} \,\tilde{<}\, \tilde{s}$, if $\tilde{r}(a) < \tilde{s}(a)$, for all $a \in K$,

(iv) $\tilde{r} \,\tilde{>}\, \tilde{s}$, if $\tilde{r}(a) > \tilde{s}(a)$, for all $a \in K$."

"**Note** Let $H$ be a non-empty set and $K$ be the non-empty set of parameters. Let $\widetilde{H}$ be the absolute soft set i.e., $F(\lambda) = H$, for all $\lambda \in K$, where $(F, K) = \widetilde{H}$.

Let $SE(\widetilde{H})$ be the collection of all soft elements of $\widetilde{H}$ and $\mathbb{R}(\mathrm{E})^*$ denote the set of all non-negative soft real numbers."

"**Definition 1.8. [7]** A mapping $\mathrm{e} : SE(\widetilde{H}) \times SE(\widetilde{H}) \to \mathbb{R}(\mathrm{E})^*$, is said to be a soft metric on soft set $\widetilde{H}$ if $\mathrm{e}$ satisfies the following conditions

(i) $\mathrm{e}(\tilde{u}, \tilde{\omega}) \,\tilde{\geq}\, \bar{0}$ for all $\tilde{u}, \tilde{\omega} \,\tilde{\in}\, \widetilde{H}$.

(ii) $\mathrm{e}(\tilde{u}, \tilde{\omega}) = \bar{0}$ if and only if $\tilde{u} = \tilde{\omega}$.

(iii) $\mathrm{e}(\tilde{u}, \tilde{\omega}) = \mathrm{e}(\tilde{\omega}, \tilde{u})$, for all $\tilde{u}, \tilde{\omega} \,\tilde{\in}\, \widetilde{H}$.

(iv) $\mathrm{e}(\tilde{u}, \tilde{\omega}) \,\tilde{\leq}\, \mathrm{e}(\tilde{u}, \tilde{\mu}) + \mathrm{e}(\tilde{\mu}, \tilde{\omega})$, for all $\tilde{u}, \tilde{\omega}, \tilde{\mu} \,\tilde{\in}\, \widetilde{H}$.

The soft set $\widetilde{H}$ with a soft metric $\mathrm{e}$ on $\widetilde{H}$ is said to be a soft metric space and is denoted be $(\widetilde{H}, \mathrm{e}, K)$ or $(\widetilde{H}, \mathrm{e})$."

“**Definition 1.9.** Let $(F, K)$ be a soft set over $H$. A soft sequence in $(F, K)$ is a function $\mathfrak{f}: \mathbb{N} \to (F, K)$ by setting $\mathfrak{f}(5) = (\tilde{u}_y, K)$ such that $(\tilde{u}_y, K)$ is a subset of $(F, K)$ for $5 \in \mathbb{N}$, and we denote it by $\{(\tilde{u}_y, K)\}$.”

“**Definition 1.10.** Let $\{\tilde{u}_y\}$ be a sequence of soft elements in a soft metric space $(\widetilde{H}, \mathrm{e})$. The sequence $\{\tilde{u}_y\}$ is said to be convergent in $(\widetilde{H}, \mathrm{e})$ if there is a soft element $\tilde{u} \,\tilde{\in}\, \widetilde{H}$ such that

$$\mathrm{e}(\tilde{u}_y, \tilde{u}) \to \bar{0} \text{ as } 5 \to \infty.$$

This means for every $\tilde{s} \,\tilde{>}\, \bar{0}$ chosen arbitrarily, there exists a natural number $N = N(\tilde{s})$, such that

$$\bar{0} \,\tilde{\leq}\, \mathrm{e}(\tilde{u}_y, \tilde{u}) \,\tilde{\leq}\, \tilde{s}, \text{ whenever } 5 > N.$$

We denote this by $\tilde{u}_y \to \tilde{u}$ as $5 \to \infty$ or by $\lim_{y \to \infty} \tilde{u}_y = \tilde{u}$. $\tilde{u}$ is said to be the limit of the sequence $\tilde{u}_y$ as $5 \to \infty$.”

“**Definition 1.11.** A sequence $\{\tilde{u}_y\}$ of soft elements in $(\widetilde{H}, \mathrm{e})$ is said to be bounded if the set $\{\mathrm{e}(\tilde{u}_m, \tilde{u}_y): m, 5 \in N\}$ of soft real numbers is bounded, i.e., there exists $\bar{M} \,\tilde{>}\, \bar{0}$ such that $\mathrm{e}(\tilde{u}_m, \tilde{u}_y) \,\tilde{\leq}\, \bar{M}$ for all $m, 5 \in N$.”

“**Definition 1.12.** A sequence $\{\tilde{u}_y\}$ of soft elements in $(\widetilde{H}, \mathrm{e})$ is said to be Cauchy sequence in $\widetilde{H}$ if corresponding to every $\tilde{s} \,\tilde{>}\, \bar{0}$ there exists $m \in N$ such that $\mathrm{e}(\tilde{u}_r, \tilde{u}_\mathbf{j}) \,\tilde{\leq}\, \tilde{s}, \forall\, r, \mathbf{j} \geq m$ i.e., $\mathrm{e}(\tilde{u}_r, \tilde{u}_\mathbf{j}) \to \bar{0}$ as $r, \mathbf{j} \to \infty$.”

“**Definition 1.13.** A soft metric space $(\widetilde{H}, \mathrm{e})$ is said to be complete if every Cauchy sequence in $\widetilde{H}$ converges to some soft element of $\widetilde{H}$. The soft metric space $(\widetilde{H}, \mathrm{e})$ is called incomplete if it is not complete.”

“**Definition 1.14.** Let $(\widetilde{H}, \mathrm{e})$ be a soft metric space. We can consider $\widetilde{H}$ as the collection of all soft elements of $\widetilde{H}$ with respect to a non-empty set of parameters $K$. Let $\mathfrak{f}: (\widetilde{H}, \mathrm{e}) \to (\widetilde{H}, \mathrm{e})$ be a mapping. If there exists a soft element $\tilde{u}_0 \,\tilde{\in}\, \widetilde{H}$ such that $\mathfrak{f}(\tilde{u}_0) = \tilde{u}_0$, then $\tilde{u}_0$ is called a fixed element of $\mathfrak{f}$.”

“**Definition 1.15.** [11] Let $(F, A)$ and $(G, B)$ be two soft sets over $H$, then the Cartesian product of $(F, A)$ and $(G, B)$ is defined as $(F, A) \times (G, B) = (S, A \times B)$, where $S: A \times B \to P(H \times H)$ and $S(a, b) = F(a) \times G(b)$ for all $(a, b) \in A \times B$.”

“**Definition 1.16.** Let $(F, A)$ and $(G, B)$ be two soft sets over $H$, then a relation from $(F, A)$ to $(G, B)$ is a soft subset of $(F, A) \times (G, B)$.”

"**Definition 1.17.** Let $(\widetilde{H}, \Theta)$ be a soft metric space. We can consider $\widetilde{H}$ as the collection of all soft elements of $\widetilde{H}$ with respect to a parameter set $K$. A mapping $P : (\widetilde{H}, \Theta) \to (\widetilde{H}, \Theta)$ is said to be a contraction mapping in $(\widetilde{H}, \Theta)$ if there is positive soft real number $\tilde{\eth}$ with $\bar{0} \widetilde{\lesssim} \tilde{\eth} \widetilde{<} \bar{1}$ such that $\Theta(P(\tilde{u}), P(\tilde{\varpi})) \widetilde{<} \tilde{\eth}\, \Theta(\tilde{u}, \tilde{\varpi})$, for all $\tilde{u}, \tilde{\varpi} \,\widetilde{\in}\, \widetilde{H}$."

"**Definition 1.18.** Let $(\widetilde{H}, \Theta)$ be a soft metric space and $P : (\widetilde{H}, \Theta) \to (\widetilde{H}, \Theta)$ a mapping. For every $u_0 \in SE(\widetilde{H})$, we can construct the sequence $\tilde{u}_y$ of soft element by choosing $u_0$ and continuing by

$$\tilde{u}_1 = P(\tilde{u}_0), \tilde{u}_2 = P(\tilde{u}_1) = P^2(\tilde{u}_0), \ldots\ldots, \tilde{u}_y = \mathrm{P}(\tilde{u}_{y-1}) = P^y(\tilde{u}_0).$$

We say that the sequence is constructed by iteration method."

## 2. Main Results

In this section, we shall prove some results in soft metric space by using binary relation.

**Definition 2.1.** Let $\widetilde{H}$ be a non-empty absolute soft set. A subset $Z$ of $\widetilde{H}^2$ is called a binary relation on $\widetilde{H}$.

For each pair $\tilde{u}, \tilde{\varpi} \in \widetilde{H}$, either one of the following conditions are satisfied

(i) $(\tilde{u}, \tilde{\varpi}) \in Z$; which can be restated as "$\tilde{u}$ is $Z$-related to $\tilde{\varpi}$" or "$\tilde{u}$ relates to $\tilde{\varpi}$ under $Z$". Sometimes, we write $\tilde{u} Z \tilde{\varpi}$ instead of $(\tilde{u}, \tilde{\varpi}) \in Z$.

(ii) $(\tilde{u}, \tilde{\varpi}) \notin Z$; which means that "$\tilde{u}$ is not $Z$-related to $\tilde{\varpi}$" or "$\tilde{u}$ does not relate to $\tilde{\varpi}$ under $Z$".

Since $\widetilde{H}^2$ and $\Phi$ are subsets of $\widetilde{H}^2$. Therefore, $\widetilde{H}^2$ and $\Phi$ are always binary relations on $\widetilde{H}$, which are respectively called the universal relation and void relation.

In this paper, $Z$ denotes non-empty binary relation, but for the sake of convenience, we use only "binary relation" instead of "non-empty binary relation".

**Definition 2.2.** Let a binary relation $Z$ defined on a non-empty absolute soft set $\widetilde{H}$ and $\tilde{u}, \tilde{\varpi} \in \widetilde{H}$. We say that $\tilde{u}$ and $\tilde{\varpi}$ are $Z$- comparative if either $(\tilde{u}, \tilde{\varpi}) \in Z$ or $(\tilde{\varpi}, \tilde{u}) \in Z$. We denote it by $[\tilde{u}, \tilde{\varpi}] \in Z$.

**Proposition 2.3.** Let $(\widetilde{H}, \Theta)$ is a soft metric space, $P$ is a self-mapping on $\widetilde{H}$, binary relation $Z$ defined on $\widetilde{H}$ and $\bar{0} \widetilde{\lesssim} \tilde{\eth} \widetilde{<} \bar{1}$ then the following contractive conditions are equivalent:

(i) $\Theta(P\tilde{u}, P\tilde{\varpi}) \leq \tilde{\eth}\, \Theta(\tilde{u}, \tilde{\varpi})$, for all $\tilde{u}, \tilde{\varpi} \in \widetilde{H}$ with $(\tilde{u}, \tilde{\varpi}) \in Z$,

(ii) $\Theta(P\tilde{u}, P\tilde{\varpi}) \leq \tilde{\eth}\, \Theta(\tilde{u}, \tilde{\varpi})$, for all $\tilde{u}, \tilde{\varpi} \in \widetilde{H}$ with $[\tilde{u}, \tilde{\varpi}] \in Z$.

**Proof** First assume that (i) holds.

Take $\tilde{u}, \tilde{\varpi} \in \widetilde{H}$ with $[\tilde{u}, \tilde{\varpi}] \in Z$.

If $(\tilde{u}, \tilde{\varpi}) \in Z$, then (ii) directly follows from (i).

Otherwise, if $(\widetilde{\omega},\widetilde{u}) \in Z$, using property of soft metric $\mathrm{e}$ and (i), we obtain

$$\mathrm{e}(P\widetilde{u},P\widetilde{\omega}) = \mathrm{e}(P\widetilde{\omega},P\widetilde{u}) \leq \tilde{\delta}\,\mathrm{e}(\widetilde{\omega},\widetilde{u}) = \tilde{\delta}\,\mathrm{e}(\widetilde{u},\widetilde{\omega}).$$

This show that (i) ⇒ (ii).

Conversely, inequality (ii) trivially implies (i).

**Definition 2.4.** Let $\widetilde{H}$ be a non-empty absolute soft set and binary relation $Z$ defined on $\widetilde{H}$.

(i) The inverse relation of $Z$, denoted by $Z^{-1}$, is defined by

$$Z^{-1} = \{(\widetilde{u},\widetilde{\omega}) \in \widetilde{H}^2 : (\widetilde{\omega},\widetilde{u}) \in Z\}.$$

(ii) Let $Z^s$ be symmetry closure of $Z$ and is defined to be the set $Z \cup Z^{-1}$. Indeed, $Z^s$ is the smallest symmetric relation on $\widetilde{H}$ containing on $\widetilde{H}$.

**Proposition 2.5.** For a binary relation $Z$ defined on a non-empty set $\widetilde{H}$,

$$(\widetilde{u},\widetilde{\omega}) \in Z^s \Leftrightarrow [\widetilde{u},\widetilde{\omega}] \in Z.$$

**Proof** Consider $(\widetilde{u},\widetilde{\omega}) \in Z^s \Leftrightarrow [\widetilde{u},\widetilde{\omega}] \in Z \cup Z^{-1}$

$$- \ (\widetilde{u},\widetilde{\omega}) \in Z \text{ or } (\widetilde{u},\widetilde{\omega}) \in Z^{-1}$$

$$- \ (\widetilde{u},\widetilde{\omega}) \in Z \text{ or } (\widetilde{\omega},\widetilde{u}) \in Z$$

$$- \ [\widetilde{u},\widetilde{\omega}] \in Z.$$

**Definition 2.6.** Let $\widetilde{H}$ be a non-empty absolute soft set and binary relation $Z$ defined on $\widetilde{H}$. A sequence $\{\widetilde{u}_y\} \subset \widetilde{H}$ is called $Z$-preserving if

$$(\widetilde{u}_y,\widetilde{u}_{y+1}) \in Z, \text{ for all } 5 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}.$$

**Definition 2.7.** Let $(\widetilde{H},\mathrm{e},K)$ be a complete soft metric space, $Z$ a binary relation defined on $\widetilde{H}$ is called $\mathrm{e}$-self closed if whenever $\{\widetilde{u}_y\}$ is $Z$-preseving sequence and

$$\widetilde{u}_y \xrightarrow{\mathrm{e}} \widetilde{u},$$

then there exists a subsequence $\{\widetilde{u}_{y_l}\}$ of $\{\widetilde{u}_y\}$ with $[\widetilde{u}_{y_l},\widetilde{u}] \in Z$ for all $l \in \mathbb{N} \cup \{0\}$.

**Definition 2.8.** Let $\widetilde{H}$ be a non-empty set and $P$ a self-mapping on $\widetilde{H}$. A binary relation $Z$ defined on $\widetilde{H}$ is called $P$- closed if for any $\widetilde{u},\widetilde{\omega} \in \widetilde{H}$,

$$(\widetilde{u},\widetilde{\omega}) \in Z \Rightarrow (P\widetilde{u},P\widetilde{\omega}) \in Z.$$

**Proposition 2.9.** Let $\widetilde{H}$ be a non-empty absolute soft set and $P$ be a self-mapping on $\widetilde{H}$. A binary relation $Z$ defined on $\widetilde{H}$ is $P$-closed, then $Z^s = Z \cup Z^{-1}$ is also $P$-closed.

**Definition 2.10.** Let $\widetilde{H}$ be a non-empty absolute soft set and binary relation $Z$ defined on $\widetilde{H}$. A subset $K$ of $\widetilde{H}$ is called $Z$-directed if for each $\widetilde{u},\widetilde{\omega} \in K$, there exists $\widetilde{\mu} \in \widetilde{H}$ such that $(\widetilde{u},\widetilde{\mu}) \in Z$ and $(\widetilde{\omega},\widetilde{\mu}) \in Z$.

**Definition 2.11.** Let $\widetilde{H}$ be a non-empty absolute soft set and binary relation $Z$ defined on $\widetilde{H}$. For $\tilde{u}, \tilde{\omega} \in \widetilde{H}$, a path of length $l$ (where $l$ is a natural number) in $Z$ from $\tilde{u}$ to $\tilde{\omega}$ is a finite sequence $\{\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_l\} \subset \widetilde{H}$ satisfying the following conditions

(i) $\tilde{\mu}_0 = \tilde{u}$ and $\tilde{\mu}_l = \tilde{\omega}$,

(ii) $(\tilde{\mu}_r, \tilde{\mu}_{r+1}) \in Z$ for each $r$ $(0 \le r \le l-1)$.

In this paper, we use the following notations

(i) $F(P) =$ the set of all fixed point of $P$,

(ii) $\widetilde{H}(P;\ Z) := \{\tilde{u} \in H: (\tilde{u}, P\tilde{u}) \in Z\}$,

(iii) $\Upsilon(\tilde{u}, \tilde{\omega}, Z) :=$ the class of all paths in $Z$ from $\tilde{u}$ to $\tilde{\omega}$

**Theorem 2.12.** Let $(\widetilde{H}, \mathrm{e}, K)$ be a complete soft metric space, binary relation $Z$ on $\widetilde{H}$ and $P$ a self-mapping on $\widetilde{H}$. Assume that the following conditions are satisfied

(i) $\widetilde{H}(P;\ Z\ )$ is non-empty,

(ii) $Z$ is $P$-closed,

(iii) Either $P$ is continuous or $Z$ is $\mathrm{e}$-self closed.

(iv) There exists $\tilde{ð} \in [0, 1)$ such that

$$\mathrm{e}(P\tilde{u}, P\tilde{\omega}) \le \tilde{ð}\ \mathrm{e}(\tilde{u}, \tilde{\omega}), \text{ for all } \tilde{u}, \tilde{\omega}\ \tilde{\in}\ \widetilde{H} \text{ with } (\tilde{u}, \tilde{\omega}) \in Z.$$

Then $P$ has a fixed point.

Moreover, if

(v) $\Upsilon(\tilde{u}, \tilde{\omega}, Z^s)$ is non-empty, for each $\tilde{u}, \tilde{\omega}\ \tilde{\in}\ \widetilde{H}$.

Then $P$ has a unique fixed point.

**Proof** Let $\tilde{u}_0$ be an arbitrary element of $\widetilde{H}(P; Z)$.

Let $\{\tilde{u}_y\}$ be defined by recursive relation $\tilde{u}_y = P^y(\tilde{u}_0)$, for all $5 \ge 0$.

As $(\tilde{u}_0, P\tilde{u}_0) \in Z$, using $(P\tilde{u}_0, P^2\tilde{u}_0)$, $(P^2\tilde{u}_0, P^3\tilde{u}_0)$, ..., $(P^y\tilde{u}_0, P^{y+1}\tilde{u}_0)$,…, $\in Z$,

so that $(\tilde{u}_y, \tilde{u}_{y+1}) \in Z$, for all $5 \in \mathbb{N}_0$.

(2.1)

Thus, the sequence $\{\tilde{u}_y\}$ is $Z$-preserving.

Applying the contractive condition (iv) to (2.1), we deduce that,

$$\mathrm{e}(\tilde{u}_{y+1}, \tilde{u}_{y+2}\ ) \le \tilde{ð}\ \mathrm{e}(\tilde{u}_y, \tilde{u}_{y+1}), \text{ for all } 5 \in \mathbb{N}_0.$$

By principle of mathematical induction, we have

$$\mathrm{e}(\tilde{u}_{y+1}, \tilde{u}_{y+2}\ ) \le \tilde{ð}^{y+1}\ \mathrm{e}(\tilde{u}_0, P\tilde{u}_0), \text{for all } 5 \in \mathbb{N}_0.$$

(2.2)

Using (2.2) and triangular inequality, for all $5 \in\ \mathbb{N}_0,\ p\ \in \mathbb{N},\ p \ge 2$, we have

$$\text{e}(\tilde{u}_{y+1},\ \tilde{u}_{y+p})\ \leq \text{e}(\tilde{u}_{y+1},\ \tilde{u}_{y+2}) + \text{e}(\tilde{u}_{y+2},\ \tilde{u}_{y+3}) + \cdots + \text{e}(\tilde{u}_{y+p-1}, \tilde{u}_{y+p})$$
$$\leq (\ \tilde{ð}^{y+1} +\ \tilde{ð}^{y+2} +\ \tilde{ð}^{y+3} + \cdots +\ \tilde{ð}^{y+p-1})\ \text{e}(\tilde{u}_0, P\tilde{u}_0)$$
$$= \ \tilde{ð}^{y}\text{e}(\ \tilde{u}_0, \tilde{\omega}_0) \textstyle\sum_{l=1}^{p-1}\ \tilde{ð}^{l}\ \ \to 0 \text{ as } 5 \to \infty,$$

this implies,

the sequence $\{\tilde{u}_y\}$ is Cauchy sequence in $\widetilde{H}$.

As $(\widetilde{H}, \text{e}, K)$ is complete soft metric space, there exists $\tilde{u}\ \in\ \widetilde{H}$ such that $\tilde{u}_y \overset{\text{e}}{\to} \tilde{u}$.

Using assumption (iii), we have

$$\tilde{u}_{y+1} = P(\tilde{u}_y) \overset{\text{e}}{\to} P(\tilde{u}).$$

As a result of uniqueness of limit, we get $P(\tilde{u}) = \ \tilde{u}$,

this implies,

$\tilde{u}$ is a fixed point of $P$.

To prove uniqueness, take $\tilde{u}, \tilde{\omega}\ \in F(P)$.

i.e., $\quad P(\tilde{u}) = \ \tilde{u} \quad$ and $\quad P(\tilde{\omega}) = \ \tilde{\omega}$.

(2.3)

By condition (v), there exists a path $(say\ \{\tilde{\mu}_0, \tilde{\mu}_1, \tilde{\mu}_2, \dots, \tilde{\mu}_l\})$ of some finite length $l$ in $Z^s$ from $\tilde{u}$ to $\tilde{\omega}$ so that

$\mu_0 = \tilde{u}, \quad \mu_l = \ \tilde{\omega}, [\tilde{\mu}_r, \tilde{\mu}_{r+1}]\ \in\ Z \quad$ for each $\quad r\ (0 \leq r \leq l-1)$.

(2.4)

As $Z$ is $P$- closed, by using Proposition 2.9, we have

$[P^y\tilde{\mu}_r,\ P^y\tilde{\mu}_{r+1}] \in\ Z$ for each $r\ (0 \leq r \leq l-1)$ and for each $5\ \in\ \mathbb{N}_0 =\ \mathbb{N}\ \cup \{0\}$.

(2.5)

Using (2.3), (2.4), (2.5), property of soft metric, condition (iv) and Proposition 2.3, we have

$$\text{e}(\tilde{u}, \tilde{\omega}) = \text{e}(P^y\tilde{\mu}_0, P^y\tilde{\mu}_l) \leq \sum_{r=0}^{l-1} \text{e}(P^y\tilde{\mu}_r, P^y\tilde{\mu}_{r+1})$$
$$\leq\ ð \textstyle\sum_{r=0}^{l-1} \text{e}(P^{y-1}\tilde{\mu}_r, P^{y-1}\tilde{\mu}_{r+1})$$
$$\leq\ ð^2 \textstyle\sum_{r=0}^{l-1} \text{e}(P^{y-2}\tilde{\mu}_r, P^{y-2}\tilde{\mu}_{r+1})$$
$$\leq$$
$$.$$
$$.$$
$$.$$
$$\leq\ ð^y \textstyle\sum_{r=0}^{l-1} \text{e}(\tilde{\mu}_r, \tilde{\mu}_{r+1})$$
$$\to 0 \text{ as } 5 \to\ \infty,$$

so that $\tilde{u} = \tilde{\varpi}$.

Thus, $P$ has a unique fixed point.

**Corollary 2.13.** Theorem 2.12 remains true if we replace condition (v) by one of the following conditions (assuming that the remaining hypotheses holds)

(i) $Z$ is complete,

(ii) $\widetilde{H}$ is $Z^s$- directed.

**Proof** If (i) holds, then for each $\tilde{u}, \tilde{\varpi} \tilde{\in} \widetilde{H}$, $[\tilde{u}, \tilde{\varpi}] \in Z$, which amounts to saying that $\{\tilde{u}, \tilde{\varpi}\}$ is a path of length $\bar{1}$ in $Z^s$ from $\tilde{u}$ to $\tilde{\varpi}$ so that $\Upsilon(\tilde{u}, \tilde{\varpi}, Z^s)$ is non-empty.

Hence, Theorem 2.12 gives rise to conclusion.

Otherwise, if (ii) holds, then for each $\tilde{u}, \tilde{\varpi} \tilde{\in} \widetilde{H}$, there exists $\tilde{\mu} \tilde{\in} \widetilde{H}$ such that $[\tilde{u}, \tilde{\mu}] \in Z$ and $[\tilde{\varpi}, \tilde{\mu}] \in Z$ so that $\{\tilde{u}, \tilde{\varpi}, \tilde{\mu}\}$ is a path from $\tilde{u}$ to $\tilde{\varpi}$ is of length $\bar{2}$ in $Z^s$.

Thus, $\Upsilon(\tilde{u}, \tilde{\varpi}, Z^s)$ is non-empty $\tilde{u}, \tilde{\varpi} \tilde{\in} \widetilde{H}$.

Again, by Theorem 2.12 also give rise to conclusion.

**Example 2.14.** Let $\widetilde{H} = \mathbb{R}$ and $\mathrm{e}(\tilde{u}, \tilde{\varpi}) = |\tilde{u} - \tilde{\varpi}|$, then $(\widetilde{H}, \mathrm{e})$ is complete soft metric space.

Consider a binary relation $Z = \{(\tilde{u}, \tilde{\varpi}) \in \mathbb{R}^2 : \tilde{u} - \tilde{\varpi} \geq \bar{0}, \tilde{u} \in \mathbb{Q}\}$ on $\widetilde{H}$ and a mapping $P : \widetilde{H} \to \widetilde{H}$ defined by

$$P(\tilde{u}) = 3 + \frac{1}{4}\tilde{u}.$$

Clearly, $\widetilde{H}(P; Z)$ is non-empty, $Z$ is $P-$closed and $P$ is continuous.

Now, for $\tilde{u}, \tilde{\varpi} \in Z$, we have

$$\begin{aligned} \mathrm{e}(P\tilde{u}, P\tilde{\varpi}) &= |(3 + \frac{1}{4}\tilde{u}) - (3 + \frac{1}{4}\tilde{\varpi})| \\ &= \frac{1}{4}|\tilde{u} - \tilde{\varpi}| \\ &= \frac{1}{4}\mathrm{e}(\tilde{u}, \tilde{\varpi}) \\ &< \frac{3}{5}\mathrm{e}(\tilde{u}, \tilde{\varpi}). \end{aligned}$$

i.e., $P$ satisfies assumption (iv) of Theorem 2.12 for $\tilde{\eth} = \frac{3}{5}$.

Hence, all the assumptions (i) - (iv) of Theorem 2.12 are holds.

Then, $P$ has a fixed point in $\widetilde{H}$.

Additionally, condition (v) of Theorem 2.12 also holds.

Therefore, $P$ has a unique fixed point $(\tilde{u} = 4)$.

**Example 2.15.** Let $\widetilde{H} = [0, 2]$ and usual soft metric $\text{ə}(\tilde{u}, \tilde{\omega}) = |\tilde{u} - \tilde{\omega}|$, then $(\widetilde{H}, \text{ə})$ is complete soft metric space. Define a binary relation $Z = \{(\tilde{0}, \tilde{0}), (\tilde{0}, \tilde{1}), (\tilde{1}, \tilde{0}), (\tilde{1}, \tilde{1}), (\tilde{0}, \tilde{2})\}$ on $\widetilde{H}$ and the mapping $: \widetilde{H} \to \widetilde{H}$ defined by

$$P(\tilde{u}) = \begin{cases} \tilde{0}, \textbf{if } 0 \leq \tilde{u} \leq 1 \\ \tilde{1}, \textbf{if } 1 < \tilde{u} \leq 2 \end{cases}$$

Clearly, $\widetilde{H}(P; Z)$ is non-empty, $P$ is not continuous and $Z$ is $P-$closed.

Take an $Z$- preserving sequence $\{\tilde{u}_y\}$ such that $\tilde{u}_y \overset{\text{ə}}{\to} \tilde{u}$,

so that $(\tilde{u}_y, \tilde{u}_{y+1}) \in Z$ for all $5 \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$.

Since sequence $\{\tilde{u}_y\}$ is $Z$-preserving.

Therefore, $(\tilde{u}_y, \tilde{u}_{y+1}) \notin \{(\tilde{0}, \tilde{2})\}$.

Now $(\tilde{u}_y, \tilde{u}_{y+1}) \in \{(\tilde{0}, \tilde{0}), (\tilde{0}, \tilde{1}), (\tilde{1}, \tilde{0}), (\tilde{1}, \tilde{1})\}$, for all $5 \in \mathbb{N}_0$, which gives to $\{\tilde{u}_y\} \,\tilde{\subset}\, \{\tilde{0}, \tilde{1}\}$.

As $\{\tilde{0}, \tilde{1}\}$ is closed, we have $[\tilde{u}_y, \tilde{u}] \in Z$.

Therefore, $Z$ is ə $-$self-closed.

All conditions of Theorem 2.12 from (i) - (iii) are satisfied.

Now we check (iv) condition,

$$\text{ə}(P\tilde{0}, P\tilde{0}) = \text{ə}(P\tilde{0}, P\tilde{1}) = \text{ə}(P\tilde{1}, P\tilde{0}) = \text{ə}(P\tilde{1}, P\tilde{1}) = \text{ə}(\tilde{0}, \tilde{0}) = \tilde{0}$$

Therefore, assumption (iv) is satisfied by taking $\tilde{\eth} = -$, for all $\tilde{u}, \tilde{\omega} \,\tilde{\in}\, \widetilde{H}$ with $(\tilde{u}, \tilde{\omega}) \in Z$.

Thus, all the assumptions (i) - (iv) of Theorem 2.12 are satisfied.

Therefore, $P$ has a fixed point in $\widetilde{H}(\tilde{u} = \tilde{0})$.

# References


1. Ahmadullah MD., Imdad M. and Gurban R., “Relation-Theoretic Metrical Fixed Point Theorems under Nonlinear Contractions”, *Fixed Point Theory,* **20** (1) (2019), 3-18.
2. Alam A. and Imdad M., “Relation- theoretic contraction principle”, *J. Fixed Point Theory Appl.,* **17** (4) (2016), 693-702.
3. Atanassov K., “Intuitionistic fuzzy sets”, *Fuzzy Sets and System,* **20** (1986), 87-9.
4. Atanassov K., “Operators over interval valued intuitionistic fuzzy sets”, *Fuzzy Sets and System,* **64** (1994), 159-174.
5. Çağman N. and Enginoglu S., “Soft set theory unit-int decision making”, *Eur. J. Oper. Res.,* **207** (2010), 845-855.

6. Das S., and Samanta S. K., "Soft real sets, soft real numbers and their properties", *J. Fuzzy Math.,* **20**(3) (2012), 551-576.
7. Das S., and Samanta S. K., "On soft metric spaces", *J. Fuzzy Maths.,* **21**(3) (2013), 707-734.
8. Feng F., *et al.,* "Soft sets combined with fuzzy sets and rough sets", *A Tentative Approach. Soft Computing,* **14** (2010), 8999-9911.
9. Gau W. I., and Buehrer D. J., "Vague sets", *IEEE Trans. Syst. Man Cybern,* **23**(2) (1993), 610-614.
10. Gorzalzany M. B., "A method of reference in approximate reasoning based on interval valued fuzzy sets", *Fuzzy Sets and System,* **21** (1987), 1-17.
11. Maji P. K., Biswas R., and Roy R., "Soft set theory", *Comput. Math. Appl.,* **45** (2003), 555-562.
12. Molodtsov D., "Soft set theory first results", *Comput. Math. Appl.,* **37** (1999), 19-31.
13. Molodtsov D., Leonov V.Y., and Kovkov D.V., "Soft sets technique and its application", *Nechetkie Sistemy i Myagkie Vychisleniya*, **1**(1) (2006), 8-39.
14. Pawlak Z., "Rough sets", *Int. J. Comput. Sci.,* **11** (1982), 341-356.
**15.** Sezgin A., and Atagn A. O., "On operations of soft sets", *Comput. Math. Appl.,* **61** (2011), 1457-1467.
16. Shabir M., and Naz M., "On soft topological spaces", *Comput. Math. Appl.,* **61** (2011), 1786-1799.
17. Zadeh L. A., "Fuzzy sets", *Inf. Control,* **8** (1965), 338-353.